An Introduction to Fuzzy Sets and Systems

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Outline

1. Fuzzy sets
2. Fuzzy logic
Fuzzy sets are sets whose elements have degrees of membership.

Fuzzy sets were introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set.

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set.
In classical sets theory, the set $H$ of real numbers from 150 to 200 is:

$$H = \{ r \in \mathbb{R} \mid 150 \leq r \leq 200 \}$$

The indicator function $\mu_H(r)$ gives the membership of each element of the universe $\mathbb{R}$ to $H$:

$$\mu_H(r) = \begin{cases} 
1 & \text{se } 150 \leq r \leq 200 \\
0 & \text{otherwise}
\end{cases}$$
Fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$.

Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.

Classical bivalent sets are in fuzzy set theory usually called crisp sets.

The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics.
Let we define the fuzzy sets $F$ of real numbers that are **close to 175** by meas the following membership function:

$$
\mu_F(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(r-175)^2},
$$
A fuzzy set $A$ in $X$ is a set of ordered pairs

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$ 

$\mu_A$ is called the membership function, $\mu_A : X \to M$, where $M$ is the membership space where each element of $X$ is mapped to.

- If $M = \{0, 1\}$, $A$ is a crisp set.
- If $\sup \mu_A = 1$, $A$ is normal. A generic fuzzy sets can be normalized by dividing $\mu_A$ for $\sup \mu_A$. 
Fuzzy Sets
Membership functions and probabilities

Suppose you had been in the desert for a week without a drink and you came upon two bottles

\[ L = \{ \text{all (potable) liquids} \} \]

\[ m_L(C \in L) = 0.91 \]

\[ Pr(A \in L) = 0.91 \]

Which would you choose to drink from first?

(figure from J.C. Bezdek and S.K. Pal, 1992)
C could contain, say, swamp water. That is membership of 0.91 means that the contents of C are fairly similar to perfectly potable liquids (e.g., pure water).

The probability that A is potable = 0.91 means that over a long run of experiments, the contents of A are expected to be potable in about 91% of the trials; in the other 9% the contents will be hydrochloric acid.
Fuzzy Sets
Membership functions and probabilities

And *after observation* of C and A?

\[
m_C(C) = 0.91 \quad \text{Pr}(A \in C) = 0
\]

(figure from J.C. Bezdek and S.K. Pal, 1992)
While it is of great intellectual interest to establish the proper connections between FL and probability, this author does not believe that doing so will change the ways in which we solve problems, because both probability and FL should be in the arsenal of tools used by engineers [Mendel, 1995].
Defined by the user, as it is need by the problem to model

(a) Triangular MF

(b) Trapezoidal MF

(c) Gaussian MF

(d) Bell MF
Jim Bezdek (1981) introduced the concept of hard and fuzzy partition in order to extend the notion of membership of pattern to clusters.

The motivation of this extension is related to the fact that a pattern often cannot be thought of as belonging to a single cluster only. In many cases, a description in which the membership of a pattern is shared among clusters is necessary.
Definition (Fuzzy number)

A fuzzy number F in a continuous universe U, e.g., a real line, is a fuzzy set F in U which is normal and convex.

Example:

$$\mu_F(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(r-175)^2},$$
Fuzzy Sets
Fuzzy singleton

Definition (Fuzzy singleton)

The singleton is a fuzzy set $A$ associated to a crisp number $x_0$. Its membership function is

$$
\mu_A(x) = \begin{cases} 
1 & \text{if } x = x_0 \\
0 & \text{otherwise}
\end{cases}
$$

[Graph showing the membership function $\mu_A(x)$ with a value of 1 at $x_0$ and 0 for other values of $x$.]
Fuzzy Sets
Linguistic variable

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A linguistic variable is characterized by a quintuple \((x, U, T(x), G, M)\) in which

- \(x\) is the name of variable;
- \(U\) is the universe of discourse;
- \(T(x)\) is the term set of \(x\), that is, the set of names of linguistic values of \(x\) with each value being a fuzzy number defined on \(U\);
- \(G\) is a syntactic rule for generating the names of values of \(x\);
- \(M\) is a semantic rule for associating with each value its meaning.
Fuzzy Sets
Linguistic variable and biomedical knowledge

- age = \{\text{very young, young, middle, old, very old}\}
- blood glucose level = \{\text{slightly increased, increased, significantly increased, strongly increased}\}.
- insulin doses = \{\text{none, low, medium, high}\}.
Definition (Union of fuzzy sets)

Let $A$ and $B$ be two fuzzy sets in $X$. The union of $A$ and $B$ is the fuzzy set $D = A \cup B$ with membership function

$$\mu_D(x) = \max\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.$$
Definition (Intersection of fuzzy sets)

Let $A$ and $B$ be two fuzzy sets in $X$. The intersection of $A$ and $B$ is the fuzzy set $D = A \cap B$ with membership function

$$
\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.
$$
Definition (Complement of a fuzzy set)

Let $A$ be a fuzzy set in $X$. The complement of $A$ in $X$ is the fuzzy set $\tilde{A}$ with membership function

$$\mu_{\tilde{A}}(x) = 1 - \mu_A(x) \quad \forall x \in X.$$
### Fuzzy Sets

Triangolar norms, t-norms (FUZZY AND)

<table>
<thead>
<tr>
<th>Fuzzy Set Operation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersection</td>
<td>( x \wedge y = \min{x, y} )</td>
</tr>
<tr>
<td>Algebraic Product</td>
<td>( x \cdot y = xy )</td>
</tr>
<tr>
<td>Bounded Product</td>
<td>( x \odot y = \max{0, x + y - 1} )</td>
</tr>
<tr>
<td>Drastic Product</td>
<td>( x \sqcap y = \begin{cases} x &amp; y = 1 \ y &amp; x = 1 \ 0 &amp; x, y &lt; 1. \end{cases} )</td>
</tr>
</tbody>
</table>

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Fuzzy Sets
Triangular co-norms (FUZZY OR)

union \hspace{2cm} \forall x \forall y = \max \{x, y\}

algebraic sum \hspace{2cm} x \oplus y = x + y - xy

bounded sum \hspace{2cm} x \ominus y = \min \{1, x + y\}

drastic sum \hspace{2cm} x \cup y = \begin{cases} x & y = 0 \\ y & x = 0 \\ 1 & x, y > 0 \end{cases}

disjoint sum \hspace{2cm} x \Delta y = \max \{\min (x, 1 - y), \\
\min (1 - x, y)\}. 

Fuzzy Sets
Geometrical interpretation [Kosko, 1991]

- *Fuzzy hypercube*

\[ \tilde{A} = \{(1, .2), (2, .7)\} \]

- Principle of Non-Contradiction: \( A \cap A' = \emptyset \)
- Principle of Excluded Middle: \( A \cup A' = U \)
Fuzzy Sets

Geometrical interpretation [Kosko, 1991]

- Fuzzy hypercube

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Fuzzy Sets
Geometrical interpretation [Kosko, 1991]

- **Fuzzy hypercube**

\[ \tilde{A} = \{(1, .2), (2, .7)\} \]

- Principle of Non-Contradiction: \( A \cap A' = \emptyset \)
- Principle of Excluded Middle: \( A \cup A' = \mathcal{U} \)
Fuzzy logic is a precise way of reasoning using imprecise information (Zadeh).

- A fuzzy if-then rule (fuzzy rule, fuzzy implication, or fuzzy conditional statement) assumes the form

\[ \text{if } x \text{ is } A \text{ then } y \text{ is } B \]  

(or \( A \rightarrow B \))

where \( A \) and \( B \) are linguistic values defined by fuzzy sets on universes of discourse \( X \) and \( Y \), respectively.

- Often "\( x \) is \( A \)" is called the antecedent or premise, while "\( y \) is \( B \)" is called the consequence or conclusion.
The linguistic knowledge on a problem is given in form of fuzzy rules

- If blood pressure is above the target and decreasing slowly then reduce drug infusion.
- If pressure is high then volume is small.
- If the road is slippery then driving is dangerous.
- If a tomato is red then it is ripe.
- If the speed is high then apply the brake a little.
A fuzzy implication $A \rightarrow B$ describes a relation between two variables $x$ and $y$;

this suggests that a fuzzy if-then rule be defined as a binary fuzzy relation $R$ on the product space $X \times Y$. 
A binary fuzzy relation $R$ is an extension of the classical Cartesian product, where each element $(x, y) \in X \times Y$ is associated with a membership grade denoted by $\mu_R(x, y)$.

Alternatively, a binary fuzzy relation $R$ can be viewed as a fuzzy set with universe $X \times Y$, and this fuzzy set is characterized by a two-dimensional membership function $\mu_R(x, y)$. 

from J-S R. Jang

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If we interpret a fuzzy implication \( F = A \rightarrow B \) as "A coupled with B" then

\[
\mu_{A \rightarrow B}(x, y) = \mu_A(x) \star \mu_B(y),
\]

where \( \star \) is a t-norm.
Fuzzy reasoning (also known as approximate reasoning) is an inference procedure used to derive conclusions from a set of fuzzy if-then rules and one or more conditions.
Derivation of $y = b$ from $x = a$ and $y = f(x)$:

- $a$ and $b$: points
- $y = f(x)$: a curve

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Fuzzy logic
Compositional rule of inference

from J-S R. Jang
To find the resulting fuzzy set B:

1. we construct a cylindrical extension $c(A)$ with base $A$ (that is, we expand the domain of $A$ from $X$ to $X \times Y$ to get $c(A)$).

$$\mu_{c(A)}(x, y) = \mu_A(x)$$

2. The intersection of $c(A)$ and $F$ forms the analog of the region of intersection $I$

$$\mu_{c(A) \cap F}(x, y) = \min[\mu_{c(A)}(x, y), \mu_F(x, y)] = \min[\mu_A(x), \mu_F(x, y)]$$

3. By projecting $c(A) \cap F$ onto the y-axis, we infer $y$ as a fuzzy set $B$ on the y-axis

$$\mu_B(y) = \max_x \{\min[\mu_A(x), \mu_F(x, y)]\} = \max_x \{\mu_A(x) \star \mu_F(x, y)\}$$

$max$-$min$ composition: $B = A \circ F$

(compositional rule of inference)
To find the resulting fuzzy set B:

- **max-min composition:** \( B = A \odot F \) (compositional rule of inference)

\[
\mu_B(y) = \max_x \min[\mu_A(x), \mu_F(x, y)] = \max_x \{\mu_A(x) \cdot \mu_F(x, y)\}
\]

- If we choose product for fuzzy AND and max for fuzzy OR, then we have **max-product composition**

\[
\mu_B(y) = \max_x [\mu_A(x) \cdot \mu_F(x, y)]
\]
Using the compositional rule of inference, we can formalize an inference procedure, called *fuzzy reasoning*, upon a set (bank) of fuzzy if-then rules.

The basic rule of inference in traditional two-valued logic is *modus ponens*, according to which we can infer the truth of a proposition $B$ from the truth of $A$ and the implication $A \rightarrow B$.
Fuzzy logic
Modus ponens (MP)

premise 1 (fact): $x$ is $A$,
premise 2 (rule): if $x$ is $A$ then $y$ is $B$,

consequence (conclusion): $y$ is $B$.

If $A$ is identified with "the tomato is red" and $B$ with "the tomato is ripe," then
if it is true that "the tomato is red," it is also true that "the tomato is ripe."
Generalized Modus Ponens (GMP) or fuzzy reasoning or approximate reasoning

**premise 1 (fact):**  
\[ x \text{ is } A', \]

**premise 2 (rule):**  
\[ \text{if } x \text{ is } A \text{ then } y \text{ is } B, \]

**consequence (conclusion):**  
\[ y \text{ is } B'. \]

where \( A' \) is close to \( A \) and \( B' \) is close to \( B \)

- If we have the same implication rule "if the tomato is red then it is ripe" and we know that "the tomato is more or less red", then we may infer that "the tomato is more or less ripe"
- **NOTE:** Generalized Modus Ponens has Modus Ponens as a special case.
Let $A$, $A'$, and $B$ be fuzzy sets of $X$, $X$, and $Y$, respectively. Assume that the fuzzy implication $A \rightarrow B$ is expressed as a fuzzy relation $R$ on $X \times Y$.

Then the fuzzy set $B'$ induced by "x is $A'$" and the fuzzy rule "if x is A then y is B" is defined by

$$\mu_{B'}(y) = \max_x \min[\mu_{(A')}(x), \mu_R(x, y)]$$

or, equivalently,

$$B' = A' \circ R = A' \circ (A \rightarrow B)$$
Fuzzy logic
Multiple Rules with Multiple Antecedents

premise 1 (fact): \( x \) is \( A' \) and \( y \) is \( B' \)
premise 2 (rule 1): if \( x \) is \( A_1 \) and \( y \) is \( B_1 \) then \( z \) is \( C_1 \)
premise 3 (rule 2): if \( x \) is \( A_2 \) and \( y \) is \( B_2 \) then \( z \) is \( C_2 \)

consequence (conclusion): \( z \) is \( C' \)
Using min and max operators as fuzzy AND and OR

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Fuzzy inference systems
Pure fuzzy inference systems

expert systems
Fuzzy inference systems
Fuzzy inference systems with fuzzifier and de-fuzzifier
Fuzzy inference systems
Fuzzy inference systems with fuzzifier and defuzzifier

Many different names:
- fuzzy inference system
- fuzzy model
- fuzzy-rule-based system
- fuzzy expert system
- fuzzy associative memory
- fuzzy logic controller
- fuzzy system

Applications: automatic control, data classification, decision analysis, expert systems, computer vision, bioinformatics
Fuzzy inference systems

- singleton

- non-singleton (fuzzy number)
Fuzzy inference systems

Defuzzifier

- Maximum Defuzzifier (MAX)
- Mean of Maxima Defuzzifier (MOM)
- Center of Area Defuzzifier (COA)
- Height Defuzzifier (HD)
Fuzzy inference systems
Mamdani fuzzy inference system

Using min and max operators as fuzzy AND and OR; singleton fuzzifier, COA defuzzifier

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Fuzzy inference systems

Rules - Air conditioning motor speed controller

- bank of rules should cover the universe of discourse
**Fuzzy inference systems**

**Rules**

- bank of rules should cover the universe of discourse
- rules are given by an expert of the application in linguistic way
- rule can be learned from data
- after learning rules can be extracted from the fuzzy system
at least $2^{215} = 32768$ different FLS’s !!:

We must decide on

- type of fuzzification (singleton or nonsingleton)
- functional forms for membership functions (triangular, trapezoidal, Gaussian, piecewise linear)
- parameters of membership functions (fixed ahead of time, tuned during a training procedure),
- composition (max-min, max-product)
- inference (minimum, product)
- defuzzifier (centroid, height, modified height).
Fuzzy inference systems
How to select "our fuzzy system?"

Criteria:
- computational cost
- empirical fit
- axiomatic force
- simplicity in modification/adaptation
- capability of learning from data
- universal function approximation property
- ...

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